

Wave turbulent statistics in non-weak wave turbulence

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Abstract

In wave turbulence, it has been believed that statistical properties are well described by the weak turbulence theory, in which nonlinear interactions among wavenumbers are assumed to be small. In the weak turbulence theory, separation of linear and nonlinear time scales derived from the weak nonlinearity is also assumed. However, the separation of the time scales is often violated even in weak turbulent systems where the nonlinear interactions are actually weak. To get rid of this inconsistency, closed equations are derived without assuming the separation of the time scales in accordance with Direct-Interaction Approximation (DIA), which has been successfully applied to Navier–Stokes turbulence. The kinetic equation of the weak turbulence theory is recovered from the DIA equations if the weak nonlinearity is assumed as an additional assumption. It suggests that the DIA equations is a natural extension of the conventional kinetic equation to *not-necessarily-weak* wave turbulence.

Keywords: wave turbulence, turbulent statistics, direct-interaction approximation

1. Introduction

Energy in wave fields such as ocean surface waves [1], ocean internal waves [2], and elastic waves on thin metal plates [3] is transferred among wavenumbers owing to nonlinear interactions. The large degree-of-freedom wave fields are called wave turbulence. The weak turbulence theory, in which the nonlinear interactions are assumed to be small, is partially successful in wave turbulent statistics. Therefore, “the *wave* turbulence” and “the *weak* turbulence” are often regarded as synonyms.

However, the separation of the linear and nonlinear time scales, which is assumed in the weak turbulence theory, is often violated as pointed out based on the weak turbulence theory [4, 5, 6]. Especially in anisotropic wave turbulence such as ocean internal waves [2] and Alfvén waves [7], the separation of the time scales is almost always violated. The violations are often observed also in

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both observations and direct numerical simulations in ocean surface waves [8] and ocean internal waves [9]. In the direct numerical simulations, the violations appear to be caused by fast non-resonant interactions. Therefore, the wave turbulence theory applicable to the non-weak wave turbulence is required for statistics of the not-necessarily-weak wave turbulence.

In Navier–Stokes turbulence which has strong nonlinear interactions among wavenumbers, direct-interaction approximation (DIA) [10] is considered to successfully describe turbulent statistics of Navier–Stokes turbulence. In this Letter, DIA is applied to a three-wave turbulent system, and a wave turbulence theory that accepts short-time and strong nonlinear interactions is constructed.

2. Weak Turbulence Theory

Wave turbulent systems with nonlinear interactions among three wavenumbers is generally governed by a canonical equation

$$i \frac{\partial a(\mathbf{p})}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\mathbf{p})} \quad (1)$$

for a complex amplitude $a(\mathbf{p})$ with Hamiltonian

$$\begin{aligned} \mathcal{H} = & (2\pi)^{-d} \int d\mathbf{p} \, \omega |a(\mathbf{p})|^2 \\ & + (2\pi)^{-2d} \int d\mathbf{p} \, d\mathbf{p}_1 d\mathbf{p}_2 \left(\mathcal{T}_{\mathbf{p}_1 \mathbf{p}_2}^{\mathbf{p}} a^*(\mathbf{p}) a(\mathbf{p}_1) a(\mathbf{p}_2) + \text{c.c.} \right), \end{aligned}$$

where \mathbf{p} is a d -dimensional wavenumber vector, and $\omega(\mathbf{p})$ is a frequency of wavenumber \mathbf{p} given by a linear dispersion relation. A matrix element $\mathcal{T}_{\mathbf{p}_1 \mathbf{p}_2}^{\mathbf{p}}$ gives strength of three-wave nonlinear interactions among \mathbf{p} , \mathbf{p}_1 and \mathbf{p}_2 . A functional derivative with respect to a is expressed by $\delta/\delta a$. Moreover, a^* denotes the complex conjugate of a , and c.c. also denotes the complex conjugate of the previous term. A d -dimensional periodic wave field in a volume $(2\pi)^d$ is considered for simplicity. Since the wave media are assumed to have spacial inversion symmetry, $\omega(\mathbf{p}) = \omega(-\mathbf{p})$ and $\mathcal{T}_{\mathbf{p}_1 \mathbf{p}_2}^{\mathbf{p}} = \mathcal{T}_{-\mathbf{p}_1 - \mathbf{p}_2}^{\mathbf{p}}$.

The canonical equation (1) is rewritten as

$$\begin{aligned} \frac{\partial a(\mathbf{p})}{\partial t} = & -i\omega(\mathbf{p})a(\mathbf{p}) \\ & - i \sum_{\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2} \mathcal{T}_{\mathbf{p}_1 \mathbf{p}_2}^{\mathbf{p}} a(\mathbf{p}_1) a(\mathbf{p}_2) - 2i \sum_{\mathbf{p}=-\mathbf{p}_1+\mathbf{p}_2} \mathcal{T}_{\mathbf{p} \mathbf{p}_1}^{\mathbf{p}_2} a^*(\mathbf{p}_1) a(\mathbf{p}_2). \end{aligned} \quad (2)$$

If the nonlinear terms in Eq. (2) is neglected, it can easily be integrated as

$$a(\mathbf{p}) = A(\mathbf{p}) \exp(-i\omega(\mathbf{p})t + i\theta(\mathbf{p})).$$

Therefore, $a(\mathbf{p})$ is called complex amplitude and represents the behaviour of a mode \mathbf{p} in the phase space.

The turbulent statistics governed by Eq. (2) is conventionally given according to the weak turbulence theory [11, 12]. Equation (2) is multiplied by $a^*(\mathbf{p}_4)$ and added complex conjugate with \mathbf{p} and \mathbf{p}_4 interchanged, and the ensemble averaging $(\overline{\cdots})$ is taken:

$$\begin{aligned} \frac{\partial n(\mathbf{p})}{\partial t} \delta_{\mathbf{p}\mathbf{p}_4} = & -i \sum_{\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2} \mathcal{T}_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}} \overline{a^*(\mathbf{p}_4)a(\mathbf{p}_1)a(\mathbf{p}_2)} \\ & - 2i \sum_{\mathbf{p}=-\mathbf{p}_1+\mathbf{p}_2} \mathcal{T}_{\mathbf{p}\mathbf{p}_1}^{\mathbf{p}_2} \overline{a^*(\mathbf{p}_4)a^*(\mathbf{p}_1)a(\mathbf{p}_2)} \\ & + \text{c.c.}(\mathbf{p} \leftrightarrow \mathbf{p}_4), \end{aligned} \quad (3)$$

where $\text{c.c.}(\mathbf{p} \leftrightarrow \mathbf{p}_4)$ denotes taking complex conjugates with \mathbf{p} and \mathbf{p}_4 interchanged. Kronecker's delta is denoted as $\delta_{\mathbf{p}_i\mathbf{p}_j}$. The wave action $n(\mathbf{p})$, which is energy of wavenumber \mathbf{p} divided by ω , is defined by $\overline{a(\mathbf{p}_i)a^*(\mathbf{p}_j)} = n(\mathbf{p}_i)\delta_{\mathbf{p}_i\mathbf{p}_j}$.

By applying the random phase approximation in the weak turbulence theory in the most primitive sense $\overline{a(\mathbf{p}_i)a^*(\mathbf{p}_j)a^*(\mathbf{p}_k)} = 0$ to the third-order correlation, the nonlinear term vanishes and

$$\frac{\partial n(\mathbf{p})}{\partial t} = 0.$$

Then, the non-zero third-order correlation is obtained by considering the time evolution of the third-order correlation. The time evolution of the third-order correlation is written as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + i\Delta\omega_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}} \right) \overline{aa_1^*a_2^*} = & -i \left(\sum_{\mathbf{p}=\mathbf{p}_3+\mathbf{p}_4} \mathcal{T}_{\mathbf{p}_3\mathbf{p}_4}^{\mathbf{p}} \overline{a_1^*a_2^*a_3a_4} + 2 \sum_{\mathbf{p}+\mathbf{p}_3=\mathbf{p}_4} \mathcal{T}_{\mathbf{p}\mathbf{p}_3}^{\mathbf{p}_4} \overline{a_1^*a_2^*a_3^*a_4} \right) \\ & + \text{c.c.}(\mathbf{p} \leftrightarrow \mathbf{p}_1) + \text{c.c.}(\mathbf{p} \leftrightarrow \mathbf{p}_2) \\ = & -2i\mathcal{T}_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}} (n_1n_2 - n(n_1 + n_2)). \end{aligned} \quad (4)$$

The random phase approximation is applied to the fourth-order correlation:

$$\begin{aligned} \overline{a_{\mathbf{p}_i}a_{\mathbf{p}_j}a_{\mathbf{p}_k}^*a_{\mathbf{p}_l}^*} = & \overline{a_{\mathbf{p}_i}a_{\mathbf{p}_k}^*} \overline{a_{\mathbf{p}_j}a_{\mathbf{p}_l}^*} + \overline{a_{\mathbf{p}_i}a_{\mathbf{p}_l}^*} \overline{a_{\mathbf{p}_j}a_{\mathbf{p}_k}^*} \\ = & n(\mathbf{p}_i)n(\mathbf{p}_j) (\delta_{\mathbf{p}_i\mathbf{p}_k}\delta_{\mathbf{p}_j\mathbf{p}_l} + \delta_{\mathbf{p}_i\mathbf{p}_l}\delta_{\mathbf{p}_j\mathbf{p}_k}). \end{aligned}$$

The frequency difference among the three wavenumbers is denoted by $\Delta\omega_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}} = \omega(\mathbf{p}) - \omega(\mathbf{p}_1) - \omega(\mathbf{p}_2)$.

The time variation of wave action n appearing in the right-hand side of Eq. (4) is possibly negligible by comparing with $1/\Delta\omega_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}}$ if the nonlinearity is weak. Here, the linear timescales are assumed to be much faster than the nonlinear time scales. The separation of the time scales is nontrivial and is often violated.

When the separation of the time scales are valid, Eq. (4) can be integrated from t_0 to $t_0 + \tau$, under the initial condition $\overline{aa_1^*a_2^*}(t_0) = 0$. The third-order correlation is obtained as

$$\overline{aa_1^*a_2^*} = \frac{-2i\mathcal{T}_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}} (n_1n_2 - n(n_1 + n_2)) (\exp(-i\Delta\omega_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}}\tau) - 1)}{-i\Delta\omega_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}}}.$$

By employing

$$\frac{i(\exp(-i\Delta\omega\tau) - 1)}{-i\Delta\omega} = \text{P.V.} \left(\frac{1}{\Delta\omega} \right) + i\pi\delta(\Delta\omega) \quad \text{as } \tau \rightarrow \infty,$$

Eq. (4) finally results in the kinetic equation:

$$\begin{aligned} \frac{\partial n}{\partial t} = 4\pi & \left(\sum_{\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2} |\mathcal{T}_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}}|^2 (n_1 n_2 - n(n_1 + n_2)) \delta(\Delta\omega_{\mathbf{p}_1\mathbf{p}_2}^{\mathbf{p}}) \right. \\ & - \sum_{\mathbf{p}_1=\mathbf{p}_2+\mathbf{p}} |\mathcal{T}_{\mathbf{p}_2\mathbf{p}}^{\mathbf{p}_1}|^2 (n_2 n - n_1(n_2 + n)) \delta(\Delta\omega_{\mathbf{p}_2\mathbf{p}}^{\mathbf{p}_1}) \\ & \left. - \sum_{\mathbf{p}_2=\mathbf{p}+\mathbf{p}_1} |\mathcal{T}_{\mathbf{p}\mathbf{p}_1}^{\mathbf{p}_2}|^2 (n n_1 - n_2(n + n_1)) \delta(\Delta\omega_{\mathbf{p}\mathbf{p}_1}^{\mathbf{p}_2}) \right). \end{aligned} \quad (5)$$

The kinetic equation indicates that the energy is transferred among wavenumbers which satisfy the resonant conditions:

$$\begin{cases} \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2, \\ \omega = \omega_1 + \omega_2, \end{cases} \quad \begin{cases} \mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}, \\ \omega_1 = \omega_2 + \omega, \end{cases} \quad \begin{cases} \mathbf{p}_2 = \mathbf{p} + \mathbf{p}_1, \\ \omega_2 = \omega + \omega_1. \end{cases}$$

3. Direct-Interaction Approximation for Wave Turbulence

As written in §2, the weak turbulence theory assumes, derived from the weak nonlinearity, that the time scales of the nonlinear energy transfer is much larger than the linear time scales. However, in almost all the anisotropic wave turbulence, the kinetic equation (5) itself gives short-time strong nonlinear interactions, and the separation of the linear and nonlinear time scales are often violated.

In this section, the direct-interaction approximation (DIA) [13], in which the separation of the time scales is not assumed but the largeness of the degrees of freedom is assumed, is applied to the wave turbulence. Note that the largeness of the degrees of freedom is also assumed implicitly in the weak turbulence theory. Instead of complex amplitude a , variables b_1 and b_2 defined by

$$b_1(\mathbf{p}) = \frac{a(\mathbf{p}) + a^*(-\mathbf{p})}{2}, \quad b_2(\mathbf{p}) = \frac{a(\mathbf{p}) - a^*(-\mathbf{p})}{2i},$$

are employed. For example, the variables are defined as

$$b_1(\mathbf{p}) = \sqrt{\frac{\sigma|\mathbf{p}|}{2\rho}} \eta(\mathbf{p}), \quad b_2(\mathbf{p}) = \sqrt{\frac{\rho}{2\sigma|\mathbf{p}|}} \phi(\mathbf{p}),$$

where $\eta(\mathbf{p})$ and $\phi(\mathbf{p})$ are the Fourier transform of the surface elevation and that of the velocity potential in deep capillary waves [11], and they are

$$b_1(\mathbf{p}) = \frac{\sqrt{\omega} N_0}{\sqrt{2g|\mathbf{k}|}} \Pi(\mathbf{p}), \quad b_2(\mathbf{p}) = -\frac{\sqrt{g}|\mathbf{k}|}{\sqrt{2\omega} N_0} \phi(\mathbf{p}),$$

where $\Pi(\mathbf{p})$ and $\phi(\mathbf{p})$ are the Fourier transform of the stratification thickness and that of the velocity potential in ocean internal waves [14]. Since $b_i(\mathbf{p})$ is the Fourier transform of real functions, $b_i(\mathbf{p}) = b_i^*(-\mathbf{p})$.

The governing equation (2) is rewritten as

$$\frac{\partial b_i(\mathbf{p})}{\partial t} = \mathcal{L}_{ij}(\mathbf{p})b_j(\mathbf{p}) + \sum_{\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2=\mathbf{0}} \mathcal{N}_{ijk}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)b_j(-\mathbf{p}_1)b_k(-\mathbf{p}_2). \quad (6)$$

The Einstein summation convention is employed. The linear and nonlinear matrix elements are

$$\begin{aligned} \mathcal{L}_{ij}(\mathbf{p}) &= (j - i)\omega(\mathbf{p}), \\ \mathcal{N}_{ijk}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2) &= \frac{i^{i+j+k}}{2} \left((-1)^{i+1} \mathcal{T}_{-\mathbf{p}_1-\mathbf{p}_2}^{\mathbf{p}} + (-1)^j \mathcal{T}_{-\mathbf{p}-\mathbf{p}_2}^{\mathbf{p}_1} + (-1)^k \mathcal{T}_{-\mathbf{p}-\mathbf{p}_1}^{\mathbf{p}_2} \right) \\ &\quad + (\text{c.c.}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2) \rightarrow (-\mathbf{p}, -\mathbf{p}_1, -\mathbf{p}_2)), \end{aligned}$$

where $\text{c.c.}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2) \rightarrow (-\mathbf{p}, -\mathbf{p}_1, -\mathbf{p}_2)$ denotes taking complex conjugates and replacing $(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)$ by $(-\mathbf{p}, -\mathbf{p}_1, -\mathbf{p}_2)$ simultaneously.

When a perturbation is added to the j th component of a wavenumber \mathbf{p}' at a time t' , which is $b_j(\mathbf{p}', t')$, the i th component of another wavenumber \mathbf{p} at a later time t , which is $b_i(\mathbf{p}, t)$, responds to the perturbation. The response function is defined as

$$G_{ij}(\mathbf{p}, t | \mathbf{p}', t') = \frac{\delta b_i(\mathbf{p}, t)}{\delta b_j(\mathbf{p}', t')}.$$

From Eq. (6), the governing equation of the response function is given by

$$\begin{aligned} \frac{\partial G_{in}(\mathbf{p}, t | \mathbf{p}', t')}{\partial t} &= \mathcal{L}_{ij}(\mathbf{p})G_{jn}(\mathbf{p}, t | \mathbf{p}', t') \\ &\quad + 2 \sum_{\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2=\mathbf{0}} \mathcal{N}_{ijk}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)b_j(-\mathbf{p}_1)G_{kn}(-\mathbf{p}_2, t | \mathbf{p}', t'). \end{aligned}$$

The initial condition of the response function is given as

$$G_{ij}(\mathbf{p}, t' | \mathbf{p}', t') = \delta_{ij}\delta(\mathbf{p} - \mathbf{p}').$$

A perturbation that removes a triad interaction among \mathbf{p}_0 , \mathbf{q}_0 and \mathbf{r}_0 that satisfies $\mathbf{p}_0 + \mathbf{q}_0 + \mathbf{r}_0 = \mathbf{0}$ is added to a wave field at a time t' . When the degrees of freedom is large enough, the effect of the perturbation is small. The variable b_i is resolved into no direct-interaction field (NDI) where the direct interaction among \mathbf{p}_0 , \mathbf{q}_0 and \mathbf{r}_0 is removed and direct-interaction field (DI). Then, $b_i = b_i^{(0)} + b_i^{(1)}$ and $|b_i^{(0)}| \gg |b_i^{(1)}|$. Similarly, the response function is also resolved into $G_{ij}^{(0)}$ in NDI and $G_{ij}^{(1)}$ in DI: $G_{ij} = G_{ij}^{(0)} + G_{ij}^{(1)}$ and $|G_{ij}^{(0)}| \gg |G_{ij}^{(1)}|$.

The equation of $b^{(0)}$ is given as

$$\frac{\partial b_i^{(0)}(\mathbf{p})}{\partial t} = \mathcal{L}_{ij}(\mathbf{p})b_j^{(0)}(\mathbf{p}) + \sum_{\substack{\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2=\mathbf{0} \\ \{\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2\} \neq \{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}}} \mathcal{N}_{ijk}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)b_j^{(0)}(-\mathbf{p}_1)b_k^{(0)}(-\mathbf{p}_2),$$

and that of $b^{(1)}$ is

$$\begin{aligned}
\frac{\partial b_i^{(1)}(\mathbf{p})}{\partial t} &= \mathcal{L}_{ij}(\mathbf{p}) b_j^{(1)}(\mathbf{p}) \\
&+ 2 \sum_{\substack{\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2=\mathbf{0} \\ \{\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2\} \neq \{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}}} \mathcal{N}_{ijk}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2) b_j^{(0)}(-\mathbf{p}_1) b_k^{(1)}(-\mathbf{p}_2) \\
&+ \mathcal{N}_{ijk}(\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0) \left(\delta_{\mathbf{p}\mathbf{p}_0} b_j^{(0)}(-\mathbf{q}_0) b_k^{(0)}(-\mathbf{r}_0) + \delta_{\mathbf{p}\mathbf{p}_0} b_j^{(0)}(\mathbf{q}_0) b_k^{(0)}(\mathbf{r}_0) \right) \\
&+ \{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}.
\end{aligned}$$

Here, $\{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}$ denotes cyclic permutations from $(\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0)$. The governing equation for $G^{(0)}$ and $G^{(1)}$ is also obtained. The solution of $b^{(1)}$ is analytically obtained

$$\begin{aligned}
b_i^{(1)}(\mathbf{p}) &= \int_{t_0}^t dt' \sum_{\mathbf{p}'} G_{in}^{(0)}(\mathbf{p}, t | \mathbf{p}', t') \mathcal{N}_{njk}(\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0) \\
&\times \left(\delta_{\mathbf{p}'\mathbf{p}_0} b_j^{(0)}(-\mathbf{q}_0, t') b_k^{(0)}(-\mathbf{r}_0, t') + \delta_{\mathbf{p}'\mathbf{p}_0} b_j^{(0)}(\mathbf{q}_0, t') b_k^{(0)}(\mathbf{r}_0, t') \right) \\
&+ \{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}
\end{aligned} \tag{7}$$

under the initial condition $b_i^{(1)}(t = t') = 0$. The solution of $G^{(1)}$ is also obtained analytically and given by $b^{(0)}$ and $G^{(0)}$.

To investigate wave turbulent statistics, the correlation between b 's defined as

$$\overline{V_{ij}(\mathbf{p}, t, t')} = \overline{b_i(\mathbf{p}, t) b_j^*(\mathbf{p}, t')}$$

is considered. Because of Eq. (6) the simultaneous correlation is governed by the following equation:

$$\begin{aligned}
\frac{\partial \overline{V_{ij}(\mathbf{p}, t, t)}}{\partial t} &= \mathcal{L}_{im}(\mathbf{p}) \overline{V_{mj}(\mathbf{p}, t, t)} + \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \mathcal{N}_{imn}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \overline{b_j(-\mathbf{p}) b_m(-\mathbf{q}) b_n(-\mathbf{r})} \\
&+ \text{c.c.}(i \leftrightarrow j).
\end{aligned} \tag{8}$$

The summation of m and n is taken over 1 and 2. DIA that removes the direct interactions among $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}$ is applied to the wave field. First, Eq. (8) is rewritten as

$$\begin{aligned}
\frac{\partial \overline{V_{ij}(\mathbf{p}, t, t)}}{\partial t} &= \mathcal{L}_{im}(\mathbf{p}) \overline{V_{mj}(\mathbf{p}, t, t)} \\
&+ \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \mathcal{N}_{imn}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \left(\overline{b_j^{(0)}(-\mathbf{p}) b_m^{(0)}(-\mathbf{q}) b_n^{(0)}(-\mathbf{r})} + \overline{b_j^{(1)}(-\mathbf{p}) b_m^{(0)}(-\mathbf{q}) b_n^{(0)}(-\mathbf{r})} \right. \\
&\left. + \overline{b_j^{(0)}(-\mathbf{p}) b_m^{(1)}(-\mathbf{q}) b_n^{(0)}(-\mathbf{r})} + \overline{b_j^{(0)}(-\mathbf{p}) b_m^{(0)}(-\mathbf{q}) b_n^{(1)}(-\mathbf{r})} \right) \\
&+ \text{c.c.}(i \leftrightarrow j),
\end{aligned} \tag{9}$$

by resolving b in the right-hand side of Eq. (8) into $b^{(0)}$ (NDI) and $b^{(1)}$ (DI). In NDI field which has no interactions among \mathbf{p} , \mathbf{q} and \mathbf{r} , $\overline{b_j^{(0)}(-\mathbf{p})b_m^{(0)}(-\mathbf{q})b_n^{(0)}(-\mathbf{r})} = 0$, since $b_j^{(0)}(\mathbf{p})$, $b_m^{(0)}(\mathbf{q})$, $b_n^{(0)}(\mathbf{r})$ are statistically independent. Second, the solutions (7) are substituted into $\overline{b_j^{(1)}(-\mathbf{p})b_m^{(0)}(-\mathbf{q})b_n^{(0)}(-\mathbf{r})}$ and similar terms. At last, Eq. (8) is expressed by the different-time correlation and the response function:

$$\begin{aligned} \frac{\partial \overline{V_{ij}(\mathbf{p}, t, t)}}{\partial t} &= \mathcal{L}_{im}(\mathbf{p}) \overline{V_{mj}(\mathbf{p}, t, t)} \\ &+ 2 \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \mathcal{N}_{imn}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \int_{t_0}^t dt' \left(\mathcal{N}_{abc}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \overline{G_{ja}(-\mathbf{p}, t | -\mathbf{p}, t')} \overline{V_{bm}(\mathbf{q}, t', t)} \overline{V_{cn}(\mathbf{r}, t', t)} \right. \\ &+ \mathcal{N}_{abc}(\mathbf{q}, \mathbf{r}, \mathbf{p}) \overline{G_{ma}(-\mathbf{q}, t | -\mathbf{q}, t')} \overline{V_{bn}(\mathbf{r}, t', t)} \overline{V_{cj}(\mathbf{p}, t', t)} \\ &+ \left. \mathcal{N}_{abc}(\mathbf{r}, \mathbf{p}, \mathbf{q}) \overline{G_{na}(-\mathbf{r}, t | -\mathbf{r}, t')} \overline{V_{bj}(\mathbf{p}, t', t)} \overline{V_{cm}(\mathbf{q}, t', t)} \right) \\ &+ \text{c.c.}(i \leftrightarrow j). \end{aligned} \quad (10)$$

The statistical independence between b_i and G_{mn} is assumed.

A similar procedure can be applied to the different-time correlation. Hence, the governing equation of the different-time correlation is obtained as

$$\begin{aligned} \frac{\partial \overline{V_{ij}(\mathbf{p}, t, t')}}{\partial t} &= \mathcal{L}_{im}(\mathbf{p}) \overline{V_{mj}(\mathbf{p}, t, t')} \\ &+ 2 \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \mathcal{N}_{imn}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \left(\int_{t_0}^{t'} dt'' \mathcal{N}_{abc}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \overline{G_{ja}(-\mathbf{p}, t' | -\mathbf{p}, t'')} \overline{V_{bm}(\mathbf{q}, t'', t)} \overline{V_{cn}(\mathbf{r}, t'', t)} \right. \\ &+ \int_{t_0}^t dt'' \left(\mathcal{N}_{abc}(\mathbf{q}, \mathbf{r}, \mathbf{p}) \overline{G_{ma}(-\mathbf{q}, t | -\mathbf{q}, t'')} \overline{V_{bn}(\mathbf{r}, t'', t)} \overline{V_{cj}(\mathbf{p}, t'', t')} \right. \\ &+ \left. \left. \mathcal{N}_{abc}(\mathbf{r}, \mathbf{p}, \mathbf{q}) \overline{G_{na}(-\mathbf{r}, t | -\mathbf{r}, t'')} \overline{V_{bj}(\mathbf{p}, t'', t')} \overline{V_{cm}(\mathbf{q}, t'', t)} \right) \right). \end{aligned} \quad (11)$$

Moreover, the governing equation of the response function is also obtained as

$$\begin{aligned} \frac{\partial \overline{G_{in}(\mathbf{p}, t | \mathbf{p}, t')}}{\partial t} &= \mathcal{L}_{ij}(\mathbf{p}) \overline{G_{jn}(\mathbf{p}, t | \mathbf{p}, t')} \\ &+ 4 \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \mathcal{N}_{ijk}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \mathcal{N}_{abc}(\mathbf{r}, \mathbf{p}, \mathbf{q}) \int_{t'}^t dt'' \overline{G_{ka}(-\mathbf{r}, t | -\mathbf{r}, t'')} \overline{G_{bn}(\mathbf{p}, t'' | \mathbf{p}, t')} \overline{V_{cj}(\mathbf{q}, t'', t)}. \end{aligned} \quad (12)$$

Equations (10–12) give a closed equation system and they are the DIA equations of the wave turbulence.

It should be emphasized that the weak nonlinearity is not assumed in the procedure. Therefore, the DIA equations of the wave turbulence (10–12) can be applied also to strongly nonlinear wave turbulent systems.

4. DIA Equations for Autocorrelation of Complex Amplitude

In the previous section, DIA is applied to the variable b . Turbulent statistics is described with the complex amplitude a to compare the weak turbulence theory in this section. The correlation of a is defined as $\overline{a(\mathbf{p}, t)a^*(\mathbf{p}, t')} = N(\mathbf{p}, t, t')$ and $\overline{a(\mathbf{p}, t)a(-\mathbf{p}, t')} = M(\mathbf{p}, t, t')$. The correlation of b , that is $\overline{V_{ij}}$ is expressed by M and N as

$$\overline{V_{ij}(\mathbf{p}, t, t')} = \frac{1}{4} \left(i^{-(i-j)} N(\mathbf{p}, t, t') + i^{i-j} N^*(-\mathbf{p}, t, t') - \left(i^{-(i+j)} M(\mathbf{p}, t, t') + i^{i+j} M^*(-\mathbf{p}, t, t') \right) \right).$$

The initial condition of the cross-correlation is that $M(\mathbf{p}, t_0, t_0) = 0$ since $a(\mathbf{p})$ and $a(-\mathbf{p})$ are uncorrelated initially. The cross-correlation at later time is much smaller than the auto-correlation, $|M(\mathbf{p}, t, t')| \ll |N(\mathbf{p}, t, t')|$. Therefore,

$$\begin{aligned} \overline{V_{ij}(\mathbf{p}, t|\mathbf{p}, t')} &= \frac{1}{4} \left(i^{-(i-j)} N(\mathbf{p}, t, t') + i^{i-j} N^*(-\mathbf{p}, t, t') \right), \\ \overline{G_{ij}(\mathbf{p}, t|\mathbf{p}, t')} &= \frac{1}{2} \left(i^{-(i-j)} G(\mathbf{p}, t, t') + i^{i-j} G^*(-\mathbf{p}, t, t') \right). \end{aligned}$$

Equation (10) is rewritten to a equation for $N(\mathbf{p}, t, t)$ as

$$\begin{aligned} \frac{\partial N(\mathbf{p}, t, t)}{\partial t} &= 2 \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \int_{t_0}^t dt' (G^*(\mathbf{p}, t, t') (|\mathcal{T}_{\mathbf{q}-\mathbf{r}}^{\mathbf{p}}|^2 N^*(-\mathbf{q}, t', t) N^*(-\mathbf{r}, t', t) \\ &+ |\mathcal{T}_{\mathbf{r}\mathbf{p}}^{-\mathbf{q}}|^2 N^*(-\mathbf{q}, t', t) N(\mathbf{r}, t', t) + |\mathcal{T}_{\mathbf{p}\mathbf{q}}^{-\mathbf{r}}|^2 N(\mathbf{q}, t', t) N^*(-\mathbf{r}, t', t)) \\ &- N(\mathbf{p}, t', t) (|\mathcal{T}_{\mathbf{q}-\mathbf{r}}^{\mathbf{p}}|^2 G(-\mathbf{q}, t, t') N^*(-\mathbf{r}, t', t) \\ &+ |\mathcal{T}_{\mathbf{r}\mathbf{p}}^{-\mathbf{q}}|^2 G(-\mathbf{q}, t, t') N(\mathbf{r}, t', t) - |\mathcal{T}_{\mathbf{p}\mathbf{q}}^{-\mathbf{r}}|^2 G^*(\mathbf{q}, t, t') N^*(-\mathbf{r}, t', t)) \\ &+ \{\mathbf{q} \leftrightarrow \mathbf{r}\}) + \text{c.c.} \end{aligned} \quad (13)$$

Similarly, the different-time correlation $N(\mathbf{p}, t, t')$ is also obtained from Eq. (11). Moreover, Eq. (12) can be expressed by N and G . Namely, the DIA equations (10–12) in the wave turbulence as equations for the correlation $\overline{V_{ij}}$ of b can be rewritten as equations for the correlation N for a .

Furthermore, the fluctuation–dissipation relation, $N(\mathbf{p}, t, t') = n(\mathbf{p}, t')G(\mathbf{p}, t, t')$, is employed. The simultaneous correlation is written as $N(\mathbf{p}, t, t) = n(\mathbf{p}, t)$ from now on. Hence, the DIA equations (10–12) can be rewritten as

$$\begin{aligned} \frac{\partial n(\mathbf{p}, t)}{\partial t} &= 2 \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \int_{t_0}^t dt' (|\mathcal{T}_{\mathbf{q}-\mathbf{r}}^{\mathbf{p}}|^2 (n(-\mathbf{q}, t')n(-\mathbf{r}, t') - n(\mathbf{p}, t') (n(-\mathbf{q}, t') + n(-\mathbf{r}, t')))) \\ &G^*(\mathbf{p}, t, t')G(-\mathbf{q}, t, t')G(-\mathbf{r}, t, t') \\ &- |\mathcal{T}_{\mathbf{r}\mathbf{p}}^{-\mathbf{q}}|^2 (n(\mathbf{p}, t')n(\mathbf{r}, t') - n(-\mathbf{q}, t') (n(\mathbf{p}, t') + n(\mathbf{r}, t'))) G^*(\mathbf{p}, t, t')G(-\mathbf{q}, t, t')G^*(\mathbf{r}, t, t') \\ &- |\mathcal{T}_{\mathbf{p}\mathbf{q}}^{-\mathbf{r}}|^2 (n(\mathbf{p}, t')n(\mathbf{q}, t') - n(-\mathbf{r}, t') (n(\mathbf{p}, t') + n(\mathbf{q}, t'))) G^*(\mathbf{p}, t, t')G^*(\mathbf{q}, t, t')G(-\mathbf{r}, t, t') \\ &+ \text{c.c.} \end{aligned} \quad (14)$$

and

$$\begin{aligned}
\frac{\partial G(\mathbf{p}, t, t')}{\partial t} &= -i\omega(\mathbf{p})G(\mathbf{p}, t, t') \\
-2\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}} \int_{t'}^t dt'' G(\mathbf{p}, t'', t') &(|\mathcal{T}_{-\mathbf{q}-\mathbf{r}}^{\mathbf{p}}|^2 (n(-\mathbf{q}, t'') + n(-\mathbf{r}, t'')) G(-\mathbf{q}, t, t'') G(-\mathbf{r}, t, t'') \\
&- |\mathcal{T}_{\mathbf{r}\mathbf{p}}^{-\mathbf{q}}|^2 (n(-\mathbf{q}, t'') - n(\mathbf{r}, t'')) G(-\mathbf{q}, t, t'') G^*(\mathbf{r}, t, t'') \\
&- |\mathcal{T}_{\mathbf{p}\mathbf{q}}^{-\mathbf{r}}|^2 (n(-\mathbf{r}, t'') - n(\mathbf{q}, t'')) G^*(\mathbf{q}, t, t'') G(-\mathbf{r}, t, t'')). \quad (15)
\end{aligned}$$

Here, $t_0 = t'$ thanks to the causality. Equations (14) and (15) are another set of the DIA equations.

Equation (14) in the DIA equations and the conventional kinetic equation look quite similar. Nonetheless, there exist some differences. Hysteresis is included in the DIA equations since the time integration is incorporated in Eq. (14). It is in contrast with the Markovian properties of the kinetic equation. Furthermore, the response functions appear in the integrand of Eq. (14). Hence, the DIA equations accept nonlinear changes of phases. The coherent structures such as freak waves which develop abruptly have the non-Markovian properties and the phase entrainment. Therefore, it is expected that the intermittency is evaluated statistically by the DIA equations.

The quadratic energy $\sum_{\mathbf{p}} \omega(\mathbf{p})n(\mathbf{p})$ is not strictly conserved for Eq. (14) since Hamiltonian which is the sum of quadratic and cubic energies is conserved. The quadratic energy is conserved for the kinetic equation (5) owing to the resonant interactions. Therefore, this non-conservation laws implies that the fast non-resonant interactions associated with strongly nonlinear coherent structures can be statistically estimated by the nonlinear parts of the response functions. The numerical simulations of Eqs. (14) and (15) are required to evaluate the fast non-resonant interactions for the specific wave turbulent systems. The quadratic momentum $\sum_{\mathbf{p}} \mathbf{p}n(\mathbf{p})$ is a conserved quantity because of the conditions of the wavenumbers. Therefore, one can find an equilibrium solution $n(\mathbf{p}) \propto (\mathbf{p} \cdot \mathbf{U})^{-1}$, where \mathbf{U} is an arbitrary constant vector. The entropy $\sum_{\mathbf{p}} \log n(\mathbf{p})$ never decrease in Eq. (14). In this manner, some of statistical natures are the same as the conventional kinetic equation (5).

For the short-time limit $t \rightarrow t_0$, Eq. (14) is rewritten as

$$\begin{aligned}
\frac{\partial n(\mathbf{p})}{\partial t} &\approx 4(t - t_0) \left(\sum_{\mathbf{p}=\mathbf{q}+\mathbf{r}} |\mathcal{T}_{\mathbf{q}\mathbf{r}}^{\mathbf{p}}|^2 (n(\mathbf{q})n(\mathbf{r}) - n(\mathbf{p})(n(\mathbf{q}) + n(\mathbf{r}))) \right. \\
&- \sum_{\mathbf{q}=\mathbf{r}+\mathbf{p}} |\mathcal{T}_{\mathbf{r}\mathbf{p}}^{\mathbf{q}}|^2 (n(\mathbf{r})n(\mathbf{p}) - n(\mathbf{q})(n(\mathbf{r}) + n(\mathbf{p}))) \\
&- \left. \sum_{\mathbf{r}=\mathbf{p}+\mathbf{q}} |\mathcal{T}_{\mathbf{p}\mathbf{q}}^{\mathbf{r}}|^2 (n(\mathbf{p})n(\mathbf{q}) - n(\mathbf{r})(n(\mathbf{p}) + n(\mathbf{q}))) \right). \quad (16)
\end{aligned}$$

This is also consistent with the short-time kinetic equation in Ref. [12]. Equation (16) is valid even for strongly nonlinear regimes since the equation is derived without employing the separation of the linear and nonlinear time scales.

5. Recovery of Kinetic Equation from DIA Equations

To recover the kinetic equation from Eq. (13), the weak nonlinearity is assumed as an “extra” assumption in this section. The time variations of the different-time correlation and the response function are as follows:

$$\begin{aligned}\frac{\partial N(\mathbf{p}, t, t')}{\partial t} &= -i\omega N(\mathbf{p}, t, t') + \text{nonlinear terms}, \\ \frac{\partial G(\mathbf{p}, t, t')}{\partial t} &= -i\omega G(\mathbf{p}, t, t') + \text{nonlinear terms}.\end{aligned}$$

The nonlinear terms of the different-time correlation and the response function do not contribute to the simultaneous correlation at the leading order. Then, by neglecting the nonlinear terms, the leading terms of the different-time correlation and the response function are obtained as

$$N(\mathbf{p}, t, t') = n(\mathbf{p}, t')e^{-i\omega(t-t')}, \quad G(\mathbf{p}, t, t') = e^{-i\omega(t-t')},$$

under the assumption of the weak nonlinearity. Therefore, since $n(\mathbf{p}, t)$ varies much slower than $1/\Delta\omega$, $n(\mathbf{p}, t) = n(\mathbf{p}, t') = n(\mathbf{p})$. This $n(\mathbf{p})$ is the very wave action in the weak turbulence theory. By substituting the leading terms of the different-time correlation and the response function into Eq. (13), we obtain

$$\begin{aligned}\frac{\partial n(\mathbf{p})}{\partial t} &= 2 \int_{t_0}^t dt' \left(\sum_{\mathbf{p}=\mathbf{q}+\mathbf{r}} |\mathcal{T}_{\mathbf{qr}}^{\mathbf{p}}|^2 (n(\mathbf{q})n(\mathbf{r}) - n(\mathbf{p})(n(\mathbf{q}) + n(\mathbf{r}))) e^{i\Delta\omega_{\mathbf{qr}}^{\mathbf{p}}(t-t')} \right. \\ &\quad - \sum_{\mathbf{q}=\mathbf{r}+\mathbf{p}} |\mathcal{T}_{\mathbf{rp}}^{\mathbf{q}}|^2 (n(\mathbf{r})n(\mathbf{p}) - n(\mathbf{q})(n(\mathbf{r}) + n(\mathbf{p}))) e^{-i\Delta\omega_{\mathbf{rp}}^{\mathbf{q}}(t-t')} \\ &\quad \left. - \sum_{\mathbf{r}=\mathbf{p}+\mathbf{q}} |\mathcal{T}_{\mathbf{pq}}^{\mathbf{r}}|^2 (n(\mathbf{p})n(\mathbf{q}) - n(\mathbf{r})(n(\mathbf{p}) + n(\mathbf{q}))) e^{-i\Delta\omega_{\mathbf{pq}}^{\mathbf{r}}(t-t')} \right) \\ &\quad + \text{c.c.}\end{aligned}\tag{17}$$

Since the separation of the time scales is assumed, t_0 can be set to $-\infty$. Then,

$$\int_{t_0}^t dt' e^{i\Delta\omega_{\mathbf{qr}}^{\mathbf{p}}(t-t')} = i(\text{P.V.} \left(\frac{1}{\Delta\omega_{\mathbf{qr}}^{\mathbf{p}}} \right) - i\pi\delta(\Delta\omega_{\mathbf{qr}}^{\mathbf{p}})).\tag{18}$$

Finally, Eq. (17) results in Eq. (5). Namely, the kinetic equation in the weak turbulence theory can be recovered by an additional assumption, that is, the weak nonlinearity to the DIA equations.

6. Concluding Remark

6.1. Discussion

In the short-time limit, Eq. (16) is consistent with what is derived along the weak turbulence theory. The conventional kinetic equation is also recovered

from the DIA equations in the weakly nonlinear limit. The nonlinear parts of the response function are irrelevant to the time variation of the wave action in both limits.

As pointed out in direct numerical simulations of four-wave weak turbulent system [15], the convergence to the kinetic equation is much faster than the convergence of the integral to the δ function in Eq. (18). During the intermediate time, which is longer than the short time and shorter than the convergence, the nonlinear parts of the response function will play a role. The role will be evaluated for specific systems with numerical simulations.

6.2. Conclusion

The closed equation system is developed for the not-necessarily-weak wave turbulence statistics according to direct-interaction approximation (DIA). In the procedure, the three assumptions below are made:

- Quantities in the field without the direct interactions are much larger than that in the perturbed field under the largeness of the degrees of freedom in the wave field.
- $b_j^{(0)}(\mathbf{p})$, $b_m^{(0)}(\mathbf{q})$ and $b_n^{(0)}(\mathbf{r})$ are statistically independent in the field without the direct interactions among \mathbf{p} , \mathbf{q} and \mathbf{r} .
- b_i and G are statistically independent.

By developing the equation system without assuming the weak nonlinearity, the DIA equations can consistently make statistical description for the wave turbulent system that has even short-time and strong nonlinear interactions. It provides us an appropriate tool to evaluate the intermittency in the wave turbulent system.

The kinetic equation in the weak turbulence theory is also recovered from the DIA equations. This indicates that the framework of DIA which harness the largeness of the degrees of freedom is the natural extension of the weak turbulence theory.

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